JOURNAL OF APPROXIMATION THEORY 4, 339–345 (1971)

Best L_q Approximate Solutions of Certain Systems of Differential Equations

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Communicated by Oved Shisha

Received March 18, 1970

INTRODUCTION

For the vector $Y = (y_1, y_2, ..., y_n)$, let $\| Y \| = \sum_{i=1}^{n} | y_i |$. Let $\{E_j\}$, $j = 1, 2, ..., n$, be the canonical basis for R^n. If $Y(x)$ is the unique solution on $[0, b]$ to

$$L[Y] \equiv Y' + F(x, Y) + G(x, Y) = h(x), \tag{1}$$

$$Y(0) = A = (a_1, a_2, ..., a_n), \tag{2}$$

then we consider best approximating $Y(x)$ by elements of the set

$$\mathscr{P}_k = \{P_k(x)\}, \tag{3}$$

where

$$P_k(x) = A + \sum_{i=1}^{k} x^i \sum_{j=1}^{n} c_{ij} E_j, \tag{4}$$

in the sense that

$$\| h(x) - L[P_k(x)] \|_q = \left\{ \int_0^b \| h(x) - L[P_k(x)] \|^q \, dx \right\}^{1/q} \tag{5}$$

is a minimum, $q \geqslant 1$.

In a previous paper in this journal [1], the author has considered the problem of approximating the solution to (1) and (2) (scalar case) by polynomials $p_k(x)$ that minimize

$$\| h(x) - L[p_k(x)] \| = \sup_{0 \leqslant x \leqslant b} | L[y(x)] - L[p_k(x)] |.$$

This paper is a sequel to Ref. [1]. Since the L_q norm is being used, the analysis is, in general, different. In the case of repetitious arguments, details will be omitted. Stein and Klopfenstein have considered a vector problem of this type; their analysis involved a linear system [4].

The Existence of Best Approximations

Showing that there exists a "vector polynomial" in the set $\mathscr{P}_k$ that is the best approximation to $Y(x)$ in the sense of (5), essentially involves proving that there exists a vector $c^* \in S \subseteq R^{(k+1)n}$, S closed, such that

$$\| h(x) - R(c^*, x)\|_q = \inf_{c \in S} \| h(x) - R(c, x)\|_q ,$$

where $R(c, x) = L[P_k(x)]$. Thus, the existence of a "vector polynomial" in $\mathscr{P}_k$ that minimizes (5) is a nonlinear best approximation problem.

In order to insure a best approximation to $Y(x)$ in sense of (5), and to insure later results, we assume that the operator L satisfies the following conditions:

(i) F and G are elements of $C[I \times R^n, R^n]$, where $I = [0, b]$.

(ii) There exist real numbers α, β and scalar functions $u(x)$, $\theta(Y)$, and $\mu(x, Y)$ such that, for all $r \geqslant 1$,

$$\| G(x, Y)\| \geqslant r^\alpha \left\| u(x)\, \theta\left(\frac{Y}{r}\right)\right\|$$

and

$$\| F(x, Y)\| \leqslant r^\beta \left| \mu\left(x, \frac{Y}{r}\right)\right|;$$

furthermore,

(a) the function $u(x)$ is not zero almost everywhere and is L_q, $q \geqslant 1$.

(b) $\theta \in C[R^n, R]$, and $\theta(Y) = 0$ if and only if $\| Y \| = 0$.

(c) $\mu(x, Y) \in C[I \times R^n, R]$.

Scalar examples of such operators may be found in Ref. [1]. An additional example is now given. Let $Y = (y_1, y_2)$, and

$$L[Y] \equiv \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \begin{bmatrix} f_1(x)\, y_1 + f_2(x)\, y_2 \\ v_1(x)\, y_1 + v_2(x)\, y_2 + g(x)(| y_1 |^3 + y_2^2 + e^{1/(y_1^2+1)}) \end{bmatrix}$$

$$= \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}, \qquad Y(0) = (a_1, a_2),$$

where f_1, f_2, v_1, v_2, g, h_1, and h_2 are continuous on $[0, b]$, and $g(x) \not\equiv 0$. Then

$$F(x, Y) = \begin{bmatrix} f_1 y_1 + f_2 y_2 \\ v_1 y_1 + v_2 y_2 \end{bmatrix}, \qquad G(x, Y) = \begin{bmatrix} 0 \\ g(|y_1|^3 + y_2^2 + e^{1/(y_1^2+1)}) \end{bmatrix}.$$

Thus, we may take $u(x) = g(x)$, $\theta(Y) = |y_1|^3 + y_2^2$, $\mu(x, Y) = \|F(x, Y)\|$, $\alpha = 2$, $\beta = 1$.

We now state the principal theorem of this section.

THEOREM 1. *Suppose that F and G satisfy conditions* (i) *and* (ii), *and that* $h(x) \in C[I, R^n]$. *If* $\alpha > \max(1, \beta)$, *then there exists a polynomial $P_k^*(x)$ in $\mathscr{P}_k$ such that*

$$\| h(x) - L[P_k^*(x)]\|_q = \inf_{P_k \in \mathscr{P}_k} \| h(x) - L[P_k(x)]\|_q .$$

The proof of this theorem is similar to that of Theorem 1 of Ref. [1] and is, therefore, omitted.

CONVERGENCE OF APPROXIMATING POLYNOMIALS

THEOREM 2. *Suppose that $Y(x)$ is the unique solution to* (1) *and* (2) *on* $[0, b]$. *Further, suppose that F, G, and h satisfy the conditions of Theorem* 1, *and that $F + G$ satisfies a Lipschitz condition in Y on every compact subset S of $I \times R^n$, i.e., if (x, Y_1) and $(x, Y_2) \in S$, then*

$$\| F(x, Y_1) + G(x, Y_1) - F(x, Y_2) - G(x, Y_2)\| \leqslant M_S \| Y_1 - Y_2 \|.$$

If $\{P_k(x)\}$ is a sequence of vector polynomials which, for each k, minimize (5), *then*

$$\lim_{k\to\infty} \| Y(x) - P_k(x)\| = 0,$$

uniformly on $[0, b]$.

Proof. For notational convenience, let $F(x, Y) + G(x, Y) = T(x, Y)$. Let $Q_k(x) \in \mathscr{P}_k$, $k = 1, 2,\ldots$, be such that

$$\| Y^{(i)}(x) - Q_k^{(i)}(x)\| \leqslant \epsilon_k , \qquad i = 0, 1, \tag{6}$$

where

$$\lim_{k\to\infty} \epsilon_k = 0. \tag{7}$$

(Such $Q_k(x)$ are known to exist.) Then

$$\begin{aligned} \| h(x) - L[Q_k(x)]\| &= \| Y' - Q_k' + T(x, Y) - T(x, Q_k)\| \\ &\leqslant \| Y' - Q_k' \| + \| T(x, Y) - T(x, Q_k)\| \\ &\leqslant \epsilon_k + M \| Y - Q_k \| \\ &\leqslant \epsilon_k(1 + M). \end{aligned} \tag{8}$$

But

$$\| h(x) - L[P_k(x)]\|_q \leqslant \| h(x) - L[Q_k(x)]\|_q .$$

Therefore

$$\lim_{k\to\infty} \| h(x) - L[P_k(x)]\|_q = 0. \tag{9}$$

Let

$$f_k(x) = h(x) - L[P_k(x)].$$

Then

$$P_k' + T(x, P_k) = h(x) - f_k(x).$$

This implies that

$$P_k(x) = H_k(x) - \int_0^x T(t, P_k(t))\, dt, \tag{10}$$

where

$$H_k(x) = \int_0^x h(t)\, dt - \int_0^x f_k(t)\, dt + A.$$

From (1) we have that $Y(x)$ is the unique solution to

$$Y(x) = H(x) - \int_0^x T(t, Y(t))\, dt, \tag{11}$$

where

$$H(x) = \int_0^x h(t)\, dt + A.$$

Also

$$\| H(x) - H_k(x)\| = \left\| \int_0^x f_k(t)\, dt \right\|. \tag{12}$$

But Hölder's inequality implies that

$$\left\| \int_0^x f_k(t)\,dt \right\| \leqslant M \|f_k(t)\|_q, \qquad \text{where } M \text{ is a constant.} \tag{13}$$

Therefore, (9) implies that

$$\lim_{k\to\infty} \left\| \int_0^x f_k(t)\,dt \right\| = 0,$$

and, consequently, by (12),

$$\lim_{k\to\infty} \| H(x) - H_k(x)\| = 0, \tag{14}$$

uniformly on $[0, b]$. Now (10), (11) and (14) suggest we consider Volterra integral equations

$$(I^* - K^*)\phi = r_n \tag{10'}$$

and

$$(I^* - K^*)\phi = r, \tag{11'}$$

where $I^*\phi = \phi$,

$$K^*\phi = \int_0^x K(x, t, \phi(t))\,dt,$$

and where

$$\lim_{n\to\infty} \| r_n(x) - r(x)\| = 0, \qquad \text{uniformly on } [0, b].$$

Suppose that $y_n(x)$ is a solution to (10′) on $[0, b]$ and that $y(x)$ is the unique solution to (11′) on $[0, b]$. Then it can be shown that if $K \in C[I \times I \times R^n, R^n]$, $I = [0, b]$, and if, for $S \subseteq I \times I \times R^n$, S compact, $(x, t, Y) \in S$, $(x, t, Z) \in S$, we have

$$\| K(x, t, Y) - K(x, t, Z)\| \leqslant M_S \| Y - Z \|,$$

then

$$\lim_{n\to\infty} \| y_n(x) - y(x)\| = 0,$$

uniformly on $[0, b]$.

Applying these remarks to (10) and (11), we have

$$\lim_{k\to\infty} \| Y(x) - P_k(x)\| = 0,$$

uniformly on $[0, b]$.

COROLLARY 1. *Suppose that the hypotheses of Theorem 2 are satisfied. Let* $Q_k(x) \in \mathscr{P}_k$, $k = 1, 2, \ldots$, *where*

$$\| Y^{(i)}(x) - Q_k^{(i)}(x) \| \leqslant \epsilon_k , \qquad i = 0, 1,$$

and where $\lim_{k\to\infty} \epsilon_k = 0$. *If* $P_k(x)$ *is a vector polynomial of degree k that minimizes* (5), *then*

$$\| Y(x) - P_k(x) \| \leqslant N\epsilon_k .$$

Proof. For all k, Theorem 2 implies that $\{(x, P_k(x))\} \subseteq S \subseteq I \times R^n$, where S is compact. Without loss of generality we may assume that $\{(x, Q_k(x))\} \subseteq S$ for all k. Then the argument following (7) shows that

$$\| f_k(x) \|_q \leqslant \epsilon_k(1 + M). \tag{15}$$

But Eqs. (10) and (11) imply that

$$Y(x) - P_k(x) = \int_0^x f_k(t)\, dt + \int_0^x T(t, P_k(t))\, dt - \int_0^x T(t, Y(t))\, dt.$$

Consequently

$$\| Y(x) - P_k(x) \| \leqslant \| f(x) \|_q + \int_0^x \| T(t, Y(t)) - T(t, P_k(t)) \|\, dt.$$

Therefore (15), the Lipschitz condition on S, and Gronwall's inequality imply that

$$\| Y(x) - P_k(x) \| \leqslant N\epsilon_k .$$

We note that Corollary 1 implies that if $Y^{(p)}(x)$ satisfies a Lipschitz condition on $[0, b]$, then

$$\| Y(x) - P_k(x) \| \leqslant \frac{N^*}{k^p}$$

for k sufficiently large.

COROLLARY 2. *Suppose that the hypotheses of Theorem 2 are satisfied. If* $P_k(x)$ *and* $Q_k(x)$, $k = 1, 2, \ldots$, *are the vector polynomials described in Corollary* 1, *and if* $\lim_{k\to\infty} k^2\epsilon_k = 0$, *then* $\lim_{k\to\infty} \| Y'(x) - P_k'(x) \| = 0$, *uniformly on* $[0, b]$.

This corollary is a direct consequence of Markoff's inequality. The details of the proof are omitted.

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Printed in Belgium